LECTURE 4 July 2, 2024

LIOUVILLE PROPERTIES.

Recall ▸ $-\Delta u = 0$ in $\mathbb{R}^n$

Harnack INEQUALITY $\exists\, C$: $\forall u \geq 0$ $\forall R > 0$ $\forall y$

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u$$

Cor. (Liouv. prop. for HARMONIC FUNCTIONS)

u bdd. above $\Rightarrow$ $u \equiv$ const.

Pf. Can assume $\inf_{\mathbb{R}^n} u = 0$. For $\varepsilon > 0$

$\exists\, x_0$: $u(x_0) \leq \varepsilon$. Harnack $\Rightarrow$

$$\sup_{B_R(x_0)} u \leq C\, u(x_0) \leq C\varepsilon$$

C indep. of R !

$\Rightarrow$ $\sup_{\mathbb{R}^n} u \leq C\varepsilon$ $\Rightarrow$ $u \equiv 0$. ∎

▸ Harnack remains true for VISCO. SOLS.

of $F(x, D^2 u) = 0$ in $\mathbb{R}^n$,

[Caffarelli-Cabré]

F UNIF. ELL. (λ, Λ), $C = C(n, \frac{\Lambda}{\lambda})$

$\Rightarrow$ Liouville property still holds.

▶ Similar results for SUBLAPLACIANS.

<u>Q</u> Is Liouville property true for SUB- or SUPERSOLUTIONS ?

No Harnack inequality!

<u>Answer 1</u> $n = 2$ $-\Delta u \le 0$, $0 \le u \le C$ in $\mathbb{R}^n$

$\Rightarrow$ $u \equiv$ const. Pf: [Protter-Weinberger]

-- not trivial. ---

<u>Answer 2</u> $\forall\, n \ge 3$ Liouv. for $-\Delta u \le 0, u \ge 0$ in $\mathbb{R}^n$

is FALSE! $u(x) = \dfrac{-1}{1+|x|^2} \in [-1, 0)$

$$\Delta u = \frac{2n + (2n-8)|x|^2}{(1+|x|^2)^3} \ge 0 \iff n \ge 4$$

For $n = 3$ $u(x) = \dfrac{-1}{\sqrt{1+|x|^2}}$ has $\Delta u \ge 0$.

<u>Answer 3</u> Similar NEGATIVE results for SUBLAPLACIANS.

Q: Is Liouville Property true for some (linear or nonlinear) $F[u] \le 0$?

Maybe LOWER ORDER TERMS can help?

A related issue: ERGODICITY of

$$dY_t = b(Y_t)\,dt + \sqrt{2}\,\sigma(Y_t)\,dW_t$$

with generator $\mathcal{L}u = \operatorname{tr}(\sigma\sigma^T D^2 u) + b \cdot Du$

$\longleftrightarrow$ $\exists$ of Lyapunov or EXAUSTION FUNCTION

$$w: \left.\begin{array}{ll} -\mathcal{L}w \ge 0 & \text{for } |x| \ge R_0 \\ w \to +\infty & \text{as } |x| \to \infty \end{array}\right\} \ (Ly)$$

Theorem: Ass. $-\mathcal{L}$ satisfies Str. Max. Princ.,

$\exists\, w: (Ly)$, $0 \le u \le C$, $-\mathcal{L}u \le 0$ in $\mathbb{R}^n$ visco sense

$\Longrightarrow$ $u \equiv$ const.

Ex. 1 Candidate $w(x) = \frac{|x|^2}{2}$: $Dw = x$, $D^2u = I$,

$$-\mathcal{L}w = \underbrace{-\operatorname{tr} A(x)}_{\le 0} - b(x)\cdot x \overset{?}{\ge} 0 \quad \text{for } |x| \ge R_0 \,?$$

Yes, if $b(x)\cdot x < 0$ & large, "b points towards 0 for large x"

Ex. 2. Ornstein – Uhlenbeck eq.: u sats.

$$-\Delta u - \underbrace{\gamma(m-x)}_{b(x)}\cdot Du \le 0 \qquad \gamma>0,\quad m\in\mathbb{R}^n$$

$A(x) = \mathrm{Id}$

$$b(x)\cdot x = \gamma(m\cdot x - |x|^2) < -n = -\operatorname{tr} A(x) \quad \text{for } |x| \text{ large}$$

$\Rightarrow$ $w = \frac{|x|^2}{2}$ sats. (L_y) $\Rightarrow$ $u \ge 0$ & bdd must be constant.

Proof of Thm.: $V_\varepsilon(x) = u(x) - \varepsilon w(x)$, $\varepsilon > 0$.

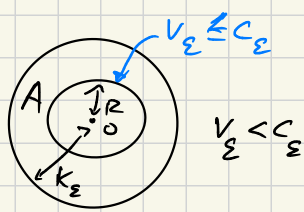
$$\forall R \ge R_0 \qquad -\mathcal{L}V_\varepsilon = \underbrace{-\mathcal{L}u}_{\le 0} + \varepsilon \underbrace{\mathcal{L}w}_{\le 0} \le 0 \qquad |x| > R.$$

Def $c_\varepsilon := \max_{|x|=R} V_\varepsilon$, $\lim_{|x|\to\infty} V_\varepsilon = -\infty$ $\Rightarrow$

$$\exists\, K_\varepsilon > R : \quad V_\varepsilon(x) < c_\varepsilon \quad \forall\, |x| > K_\varepsilon$$

Weak Max Princ. in A

$$\max_A V_\varepsilon \le \max_{\partial A} V_\varepsilon = c_\varepsilon$$



$$\Rightarrow \forall\, |x| \ge R \qquad V_\varepsilon(x) \le c_\varepsilon \le \max_{|x|=R} u - \varepsilon \min_{|x|=R} w$$

Let $\varepsilon \searrow 0$ $\Rightarrow$ $u(x) \le \max_{|x|=R} u \quad \forall\, |x| \ge R$

Weak Max Princ. in $B_R(0)$ $\Rightarrow u(x) \le \max_{|x|=R} u$ $|x|\le R$

$\Rightarrow$ $\forall x \in \mathbb{R}^n$ $u(x) \le \max_{|x|=R} u$.

Strong Max Princ. $\Rightarrow$ $u \equiv$ const. ▨

Rmks ▶ $-\Delta u \le 0$ in $\mathbb{R}^2$: $w(x) = \log|x|$ sats (Ly)

Then $0 \le u \le C$ must be constant. $\left[\limsup_{|x|\to\infty} \frac{u}{w} \le 0\right]$ is enough

▶ If $n \ge 3$ Lyap. fn. for $-\Delta$ $\cancel{\exists}$ ▨

For Nonlinear eqs. $F(x, u, Du, D^2u) \le 0$!

"Abstract Theorem" (B.-Cesaroni, B-Goffi). Assume:

(i) F proper & CONT., $F(x, r, 0, 0) \ge 0$ $\forall r$

(S.Ad.) $F[\varphi - \psi] \le F[\varphi] - F[\psi]$ $\forall \varphi, \psi \in C^2$

(ii) F sats. weak Comp. Princ. on bdd. sets

(iii) $\exists$ Lyap. fn. w $\left.\begin{array}{ll} F[w] \ge 0 & |x| > R_0 \\ \lim_{|x|\to\infty} w(x) = +\infty & \end{array}\right\}$ (Ly)

(iv) F sats STRONG MAX PRINCIPLE

Then $u \in USC(\mathbb{R}^n)$, $u \ge 0$, $F[u] \le 0$ in $\mathbb{R}^n$,

$\limsup_{|x|\to\infty} \frac{u}{w} \le 0$ $\Rightarrow$ $u \equiv$ const.

Proof: Along the lines of the linear case ... ∎

CONCRETE EXAMPLES:

- For Strong M.P.: previous lecture
- For Weak Comp. Princ.: see [CIL] etc...
- Sub. Additivity: ok for Bellman eqs.

$$F[u] = \inf_\alpha L^\alpha u \quad , \quad L^\alpha u = -\operatorname{tr} A^\alpha D^2 u - b^\alpha \cdot Du + c^\alpha u$$

- w Lyap. fn? $\quad w(x) = \frac{|x|^2}{2}$ supersol of $F[w] \geq 0$

if $\quad \sup_\alpha \left\{ \operatorname{tr} A^\alpha(x) + b^\alpha(x) \cdot x - c^\alpha(x) \frac{|x|^2}{2} \right\} \leq 0 \quad |x| > R$

▶ ok if $c^\alpha(x) \geq c_0 > 0$, $|b^\alpha| \leq o(|x|)$, $|A^\alpha| \leq o(|x|^2)$

or

▶ ok if $c^\alpha \geq 0$, $\limsup_{|x| \to \infty} \sup_\alpha \left\{ \operatorname{tr} A^\alpha(x) + b^\alpha(x) \cdot x \right\} < 0$

as in ORNSTEIN UHLENBECK

"RECURRENSE ASS." or "DISSIPATIVITY".

- can give SHARP EXPLICIT CONDITIONS for Liouv. Prop.

if $\quad \mathcal{M}^-(D^2 u) + H(x, u, Du) \leq 0$

& for

$$\mathcal{M}^-((D^2_{\mathcal{X}} u)^+) + H(x, u, D_{\mathcal{X}} u) \leq 0$$

$\mathcal{X}$ satisfying (HC).

SOME APPLICATIONS :

- Regularity theory via blow-up method:

  Gidas – Spruck ~ 1981

- "critical value" of F : Find $c \in \mathbb{R}$: $\exists$ visco sol.

  of $\quad F(x, v, Dv, D^2 v) = c \quad$ in $\mathbb{R}^n$

  $\longleftrightarrow$ ergodicity of controlled diffusions

- long time behavior of $u_t + F(x, Du, D^2 u) = 0$:

$$\lim_{t \to +\infty} u(x,t) \overset{?}{=} c$$